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# Inference on Modelling Cross-Sectional Dependence for a Varying-Coefficient Model

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## Abstract

In this note, I have studied a vary-coefficient model under cross-sectional dependence. The technique of Robinson (2011) is employed to mimic the dependence among cross-sectional data sets. The asymptotic normality is established for the proposed estimator.

*Keywords:* Asymptotic theory; cross-sectional dependence; varying-coefficient.

*JEL classification:* C13, C14, C51

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# 1 Introduction

The cross sectional dependence has been a hot topic for the past two decades. A dominant branch of modelling the cross-sectional dependence is to use a factor structure in panel data models (c.f. Pesaran (2006), Bai (2009) and so forth). Recently, Robinson (2011) and Lee and Robinson (2016) have employed the time series technique to model the dependence among cross-sectional data sets. Following the spirit of their work, I consider a varying-coefficient model with cross-sectional dependence in this study.

## 2 Model Specification

The mode is as follows:

$$y_i = x_i' \beta(z_i) + u_i. \quad (2.1)$$

$z_i \in [0, 1]$  is the so-called univariate index variable (Wang and Xia (2009)) and  $x_i$  is a  $p \times 1$  vector. For simplicity, we consider the scalar case for  $z_i$  only and it is straightforward to extend  $z_i$  to multivariate case. To distinguish  $x_i$  and  $z_i$ , they are referred to as regressors and covariates hereafter. In order to impose the cross-sectional dependence, we follow Robinson (2011) and Lee and Robinson (2016) and denote that

$$u_i = \sigma(x_i, z_i) e_i, \quad e_i = \sum_{j=1}^{\infty} b_{ij} \varepsilon_j, \quad b_{ii} \neq 0, \quad B_i = \sum_{j=1}^{\infty} b_{ij}^2 < \infty \quad \text{for } i = 1, \dots, N, \quad (2.2)$$

where  $\sigma : \mathbb{R}^p \times [0, 1] \rightarrow \mathbb{R}$ , the  $b_{ij}$  are real constants, and  $\{\varepsilon_j, j \geq 1\}$  is a sequence of independent random variables with zero mean and unit variance, independent of  $\{x_j, j \geq 1\}$  and  $\{z_j, j \geq 1\}$ .

**Remark:**

Notice  $b_{ii} \neq 0$  rules out the case where the error term  $e_i$  does not change across index  $i$ . For example, without the restriction of  $b_{ii} \neq 0$ , one can let  $\sigma(x_i, z_i) = 1$ ,  $b_{i1} = 1$ ,  $b_{ij} = 0$  for  $i = 1, \dots, N$  and  $j = 2, \dots, \infty$ . Then the model will reduce to  $y_i = x_i' \beta(z_i) + \varepsilon_1$ . In this case, the consistent estimation cannot be achieved at all.

In this note, our kernel function is denoted as:

$$K_h(z_i - z) = \frac{1}{h} K\left(\frac{z_i - z}{h}\right), \quad (2.3)$$

where  $K(w)$  is symmetric denoted on  $[-1, 1]$  satisfying  $\int_{-1}^1 K(w) dw = 1$  and  $h$  is the bandwidth. In order to facility the development, we adopt the following assumptions.

**Assumptions:**

1.  $\{\varepsilon_j, j \geq 1\}$  is a sequence of independent random variables with zero mean and unit variance, independent of  $\{x_j, j \geq 1\}$  and  $\{z_j, j \geq 1\}$ .  $E[\varepsilon_j] = 0$ ,  $E[\varepsilon_j^2] = 1$  and  $\max_{j \geq 1} E[\varepsilon_j^{2+\nu}] < \infty$ .  $\sigma^2(x, z)$  is a uniformly bounded. Moreover,  $\max_{1 \leq i \leq N} E\|x_i\|^4 < \infty$  and  $\max_{z \in [0,1]} \|\beta(z)\| < \infty$ .
2. Let  $E[x_i x_i' | z_i = z] = \Sigma_{xi}(z)$ , where  $\|\Sigma_{xi}(z)\|$  is uniformly bounded on  $[0, 1]$ .  $\Sigma_{xi}(z)$  has bounded continuous second order derivative with respect to  $z$  uniformly in  $i$ . Moreover,  $x_i$  is the function of  $z_i$  and independent of  $z_j$  for  $i \neq j$ .
3. For  $1 \leq i \neq j \leq N$ , let  $f_{ij}(w_1, w_2)$  denote the joint density function for  $(z_i, z_j)$  and be bounded uniformly in  $i, j$ . For  $i = 1, \dots, N$ , let  $f_i(w)$  denote the density function for  $z_i$  and be bounded uniformly in  $i$ . In addition,  $f_i(w)$  has uniformly bounded continuous second order derivative with respect to  $w$ .
4. (a)  $Nh \rightarrow \infty$ ,  $h \rightarrow 0$ ;  
 (b)  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N B_i = B$  and  $\max_{1 \leq i \leq N} |B_i| \leq C_1$ , where  $C_1$  is a constant. Also, for  $\forall z \in [0, 1]$ , let  $V_2(z) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \Sigma_{xi}(z) f_i(z)$  be positive definite uniformly in  $z$ .  
 (c)  $\max_{1 \leq j \leq N} \frac{1}{\sqrt{Nh}} \sum_{i=1}^N |b_{ij}| \rightarrow 0$ ;  
 (d)  $\frac{\Delta_{2N}}{N^2} \rightarrow 0$  and  $\frac{\sqrt{\Delta_{1N}}}{Nh} \rightarrow 0$ , where

$$\Delta_{1N} = \sum_{i,j=1, i \neq j}^N \iint |f_{ij}(w_1, w_2) - f_i(w_1)f_j(w_2)| dw_1 dw_2,$$

$$\Delta_{2N} = \sum_{i,j=1, i \neq j}^N |\gamma_{i,j}|, \quad \gamma_{i,j} = \text{Cov}(e_i, e_j);$$

Assumptions 1-4 are standard in the literature (c.f. Wang and Xia (2009), Lee and Robinson (2016)), so the relevant discussions are omitted. In Assumption 4.c,  $\max_{1 \leq j \leq N} \frac{1}{\sqrt{Nh}} \sum_{i=1}^N |b_{ij}| \rightarrow 0$  certainly captures the i.i.d. case. For example, let  $\sigma(x_i, z_i) = 1$ . When  $u_i$  is i.i.d., the matrix  $\mathcal{B} = \{b_{i,j}\}_{N \times N} = I_N$ . Then it is easy to see that  $\max_{1 \leq j \leq N} \frac{1}{\sqrt{Nh}} \sum_{i=1}^N |b_{ij}| \rightarrow 0$  holds. Notice that if  $z_i$  is independent across  $i$ , one can easily show that  $\Delta_{1N} = 0$  and  $\gamma_{i,j} = 0$ , so Assumption 4.d holds immediately.

For any given  $z \in [0, 1]$ , we investigate the next estimator.

$$\hat{\beta}(z) = \left( \sum_{i=1}^N x_i x_i' K_h(z_i - z) \right)^{-1} \sum_{i=1}^N x_i y_i K_h(z_i - z). \quad (2.4)$$

Then the next result follows based on the above settings.

**Theorem 2.1.** *Under Assumptions 1-4,*

$$\sqrt{Nh} \left( \hat{\beta}(z) - \beta(z) - O_P(h^2) \right) \rightarrow_D N(0, V_2^{-1}(z) V_1(z) V_2^{-1}(z))$$

where

$$\begin{aligned} V_1(z) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N f_i(z) B_i \int \Sigma_{xx\sigma}(w) K^2(w) dw \\ &\quad + \lim_{N \rightarrow \infty} \frac{h}{N} \sum_{i_1=1}^N \sum_{i_2=1}^N \gamma_{i_1, i_2} f_{i_1}(z) f_{i_2}(z) \tilde{\eta} \tilde{\eta}', \end{aligned}$$

where  $\tilde{\eta} = \int \eta(w) K(w) dw$ ,  $\eta(z_i) = E[x_i \sigma(x_i, z_i) | z_i]$ ,  $\Sigma_{xx\sigma}(z_i) = E[x_i x_i' \sigma^2(x_i, z_i) | z_i]$  and  $V_2(z)$  is denoted in Assumption 4.

### 3 Conclusion

In this note, I have studied a vary-coefficient model under cross-sectional dependence. The technique of Robinson (2011) and Lee and Robinson (2016) is employed to mimic the dependence among cross-sectional data sets. The asymptotic normality is established for the proposed estimator. The optimal bandwidth selection has been achieved under i.i.d. case in Li and Racine (2010), but what the optimal bandwidth looks like under cross-sectional dependence remains unsolved.

### Appendix

**Lemma A.1.** *Let  $\zeta_i = (x_i, z_i)$  and Assumption 3 hold. For any bounded function  $g(w)$  with  $w = (w_1, w_2) \in \mathbb{R}^p \times [0, 1]$  having that  $E[g(\zeta_i)g(\zeta_j)]$  with  $i \neq j$  and  $E[g(\zeta_i)]$  exist uniformly in  $1 \leq i, j \leq N$ , we obtain that*

$$\left| \sum_{i,j=1, i \neq j}^{\infty} \{E[g(\zeta_i)g(\zeta_j)] - E[g(\zeta_i)] E[g(\zeta_j)]\} \right| = O(\Delta_{1N}). \quad (\text{A.1})$$

**Proof of Lemma A.1:**

$$\begin{aligned} &\left| \sum_{i,j=1, i \neq j}^{\infty} \{E[g(\zeta_i)g(\zeta_j)] - E[g(\zeta_i)] E[g(\zeta_j)]\} \right| \\ &= \left| \sum_{i,j=1, i \neq j}^{\infty} \int g(w_1)g(w_2) (f_{ij}(w_1, w_2) - f_i(w_1)f_j(w_2)) dw_1 dw_2 \right| \\ &= O(1) \sum_{i,j=1, i \neq j}^{\infty} \int |f_{ij}(w_1, w_2) - f_i(w_1)f_j(w_2)| dw_1 dw_2 = O(\Delta_{1N}). \end{aligned} \quad (\text{A.2})$$

Then the proof is complete. ■

**Lemma A.2.** *Under Assumptions 1-4, for any given  $z \in [0, 1]$*

1.  $\frac{1}{N} \sum_{i=1}^N x_i x'_i K_h(z_i, z) - \frac{1}{N} \sum_{i=1}^N \Sigma_{xi}(z) f_i(z) = O_P(h^2) + O_P\left(\frac{\sqrt{\Delta_{1N}}}{Nh}\right);$
2.  $\frac{1}{N} \sum_{i=1}^N x_i u_i K_h(z_i, z) = O_P\left(\frac{1}{\sqrt{Nh}}\right) + O_P\left(\frac{\sqrt{\Delta_{1N}}}{Nh}\right) + O_P\left(\frac{\sqrt{\Delta_{2N}}}{N}\right);$
3.  $\frac{1}{N} \sum_{i=1}^N x_i x'_i (\beta(z_i) - \beta(z)) K_h(z_i - z) = O_P(h^2);$

**Proof of Lemma A.2:**

1) Write

$$\begin{aligned}
& E \left[ \frac{1}{N} \sum_{i=1}^N x_i x'_i K_h(z_i - z) \right] = \frac{1}{N} \sum_{i=1}^N E [x_i x'_i K_h(z_i - z)] \\
&= \frac{1}{N} \sum_{i=1}^N \int \Sigma_{xi}(w) K_h(w - z) f_i(w) dw \\
&= \frac{1}{N} \sum_{i=1}^N \int \Sigma_{xi}(z + hw) K_h(w) f_i(z + hw) dw \\
&= \frac{1}{N} \sum_{i=1}^N \Sigma_{xi}(z) f_i(z) + O(h^2), \tag{A.3}
\end{aligned}$$

where the fourth equality follows from using Taylor expansion on each element of  $\Sigma_{xi}(w)$  and  $f_i(w)$ .

For the second moment, write

$$\begin{aligned}
& E \left\| \frac{1}{N} \sum_{i=1}^N (x_i x'_i K_h(z_i - z) - \Sigma_{xi}(z) f_i(z)) \right\|^2 \\
&= \frac{1}{N^2} \sum_{m=1}^p \sum_{n=1}^p \sum_{i=1}^N E [x_{i,m} x_{i,n} K_h(z_i - z) - \Sigma_{xi,mn}(z) f_i(z)]^2 \\
&+ \frac{1}{N^2} \sum_{m=1}^p \sum_{n=1}^p \sum_{i,j=1, i \neq j}^N E \left[ (x_{i,m} x_{i,n} K_h(z_i - z) - \Sigma_{xi,mn}(z) f_i(z)) \right. \\
&\quad \cdot (x_{j,m} x_{j,n} K_h(z_j - z) - \Sigma_{xi,mn}(z) f_j(z)) \left. \right] \\
&:= A_1 + A_2, \tag{A.4}
\end{aligned}$$

where  $\Sigma_{xi,mn}(z)$  denotes the  $(m, n)^{th}$  element of  $\Sigma_{xi}(z)$  for  $i = 1, \dots, N$ .

For  $A_1$ , write

$$\begin{aligned}
A_1 &= \frac{1}{N^2} \sum_{m=1}^p \sum_{n=1}^p \sum_{i=1}^N E [x_{i,m} x_{i,n} K_h(z_i - z) - \Sigma_{xi,mn}(z) f_i(z)]^2 \\
&\leq \frac{1}{N^2 h} \sum_{m=1}^p \sum_{n=1}^p \sum_{i=1}^N E [x_{i,m}^2 x_{i,n}^2 K_h(z_i - z)] \\
&\leq \frac{1}{N^2 h} \sum_{m=1}^p \sum_{n=1}^p \sum_{i=1}^N E [x_{i,m}^2 x_{i,n}^2 K_h(z_i - z)] = O\left(\frac{1}{Nh}\right), \tag{A.5}
\end{aligned}$$

where we have used the uniform boundedness of  $K(w)$ .

For  $A_2$ , write

$$\begin{aligned}
& \sum_{i,j=1, i \neq j}^N E \left[ (x_{i,m} x_{i,n} K_h(z_i - z) - \Sigma_{xi,mn}(z) f_i(z)) \right. \\
& \quad \cdot (x_{j,m} x_{j,n} K_h(z_j - z) - \Sigma_{xj,mn}(z) f_j(z)) \left. \right] \\
&= \sum_{i,j=1, i \neq j}^N E \left[ (\Sigma_{xi,mn}(z_i) K_h(z_i - z) - \Sigma_{xi,mn}(z) f_i(z)) \right. \\
& \quad \cdot (\Sigma_{xj,mn}(z_j) K_h(z_j - z) - \Sigma_{xj,mn}(z) f_j(z)) \left. \right] \\
&= \sum_{i,j=1, i \neq j}^N \iint (\Sigma_{xi,mn}(w_1) K_h(w_1 - z) - \Sigma_{xi,mn}(z) f_i(z)) \\
& \quad \cdot (\Sigma_{xj,mn}(w_2) K_h(w_2 - z) - \Sigma_{xj,mn}(z) f_j(z)) f_{ij}(w_1, w_2) dw_1 dw_2 \\
&= \sum_{i,j=1, i \neq j}^N \int (\Sigma_{xi,mn}(w_1) K_h(w_1 - z) - \Sigma_{xi,mn}(z) f_i(z)) f_i(w_1) dw_1 \\
& \quad \cdot \int (\Sigma_{xj,mn}(w_2) K_h(w_2 - z) - \Sigma_{xj,mn}(z) f_j(z)) f_j(w_2) dw_2 \\
& \quad + \sum_{i,j=1, i \neq j}^N \iint (\Sigma_{xi,mn}(w_1) K_h(w_1 - z) - \Sigma_{xi,mn}(z) f_i(z)) \\
& \quad \cdot (\Sigma_{xj,mn}(w_2) K_h(w_2 - z) - \Sigma_{xj,mn}(z) f_j(z)) (f_{ij}(w_1, w_2) - f_i(w_1) f_j(w_2)) dw_1 dw_2 \\
&\leq O(h^4 N^2) + \frac{1}{h^2} \sum_{i,j=1, i \neq j}^N \iint |f_{ij}(w_1, w_2) - f_i(w_1) f_j(w_2)| dw_1 dw_2 \\
&\leq O(h^4 N^2) + O\left(\frac{\Delta_{1N}}{h^2}\right), \tag{A.6}
\end{aligned}$$

where the first inequality follows from (A.3), uniform boundedness of  $\Sigma_{xi}(w_1, w_2)$  and  $K(w)$ ; the second inequality follows from Assumption 5.

Thus, we have  $A_2 = O(h^4) + O\left(\frac{\Delta_{1N}}{N^2 h^2}\right)$ . Based on the above, the first result of this lemma follows.

2) It is easy to know that  $E\left[\frac{1}{N} \sum_{i=1}^N x_i u_i K_{H,\Theta}(z_i, z)\right] = 0$ . For the second moment, write

$$\begin{aligned}
& E \left\| \frac{1}{N} \sum_{i=1}^N x_i u_i K_{H,\Theta}(z_i, z) \right\|^2 \\
&= \frac{1}{N^2} \sum_{i=1}^N E [\|x_i\|^2 \sigma^2(x_i, z_i) e_i^2 K_h^2(z_i - z)] \\
& \quad + \frac{1}{N^2} \sum_{i,j=1, i \neq j}^N E [x'_i x_j \sigma(x_i, z_i) \sigma(x_j, z_j) K_h(z_i, -z) K_h(z_j - z)] E[e_i e_j] \\
&\leq O(1) \frac{1}{N^2 h} \sum_{i=1}^N E [\|x_i\|^2 \sigma^2(x_i, z_i) K_h(z_i - z)] E[e_i^2] \\
& \quad + \frac{1}{N^2} \sum_{i,j=1, i \neq j}^N E [x'_i x_j \sigma(x_i, z_i) \sigma(x_j, z_j) K_h(z_i, -z) K_h(z_j - z)] E[e_i e_j]
\end{aligned}$$

$$:= A_1 + A_2. \quad (\text{A.7})$$

For  $A_1$ , it is easy to show that

$$\begin{aligned} A_1 &= O(1) \frac{1}{N^2 h} \sum_{i=1}^N E [\|x_i\|^2 \sigma^2(x_i, z_i) K_h(z_i - z)] E[e_i^2] \\ &\leq O(1) \frac{1}{N^2 h} \sum_{i=1}^N E [\|x_i\|^2 \sigma^2(x_i, z_i) K_h(z_i - z)] \\ &\leq O(1) \frac{1}{N^2 h} \sum_{i=1}^N E [\|x_i\|^2 K_h(z_i - z)] = O\left(\frac{1}{Nh}\right). \end{aligned}$$

For  $A_2$ , write

$$\begin{aligned} &\sum_{i,j=1, i \neq j}^N \left| E [x_i' x_j \sigma(x_i, z_i) \sigma(x_j, z_j) K_h(z_i - z) K_h(z_j - z)] E[e_i e_j] \right| \\ &= \sum_{i,j=1, i \neq j}^N \left| E [\eta_i(z_i)' \eta_j(z_j) K_h(z_i - z) K_h(z_j - z)] \gamma_{i,j} \right| \\ &= \sum_{i,j=1, i \neq j}^N \left| \iint \eta_i(w_1)' K_h(w_1 - z_c) \eta_j(w_2) K_h(w_2 - z_c) f_{ij}(w_1, w_2) dw_1 dw_2 \gamma_{i,j} \right| \\ &\leq \sum_{i,j=1, i \neq j}^N \left| \int \eta_i(w_1)' K_h(w_1 - z_c) f_i(w_1) dw_1 \int \eta_j(w_2) K_h(w_2 - z_c) f_j(w_2) dw_2 \cdot \gamma_{i,j} \right| \\ &\quad + \sum_{i,j=1, i \neq j}^N \left| \iint \eta(w_1)' K_h(w_1 - z) \eta(w_2) K_h(w_2 - z) \right. \\ &\quad \cdot \left. (f_{ij}(w_1, w_2) - f_i(w_1) f_j(w_2)) dw_1 dw_2 \cdot \gamma_{i,j} \right| \\ &\leq O(1) \sum_{i,j=1, i \neq j}^N |\gamma_{i,j}| + \frac{1}{h^2} \sum_{i,j=1, i \neq j}^N \iint |f_{ij}(w_1, w_2) - f_i(w_1) f_j(w_2)| dw_1 dw_2 \\ &\leq O(\Delta_{2N}) + O\left(\frac{\Delta_{1N}}{h^2}\right), \end{aligned} \quad (\text{A.8})$$

where the second inequality follows from the uniform boundedness on  $\eta(\cdot)$  and  $f_i(w)$ .

Therefore, for  $A_2$ , we obtain  $A_2 = O\left(\frac{\Delta_{2N}}{N^2}\right) + O\left(\frac{\Delta_{1N}}{N^2 h^2}\right)$ . Based on the analysis on  $A_1$  and  $A_2$ , the result follows.

3) We then focus on  $\frac{1}{N} \sum_{i=1}^N x_i x_i' (\beta(z_i) - \beta(z)) K_h(z_i - z)$ .

$$\begin{aligned} &E \left\| \frac{1}{N} \sum_{i=1}^N x_i x_i' (\beta(z_i) - \beta(z)) K_h(z_i - z) \right\| \\ &\leq \frac{1}{N} \sum_{i=1}^N E [\|x_i x_i' (\beta(z_i) - \beta(z))\| K_h(z_i - z)] \\ &\leq O(1) \frac{1}{N} \sum_{i=1}^N \int K_h(w - z) f_i(w) dw = O(h^2), \end{aligned}$$

where the last line follows from (A.3). Then, the result follows immediately. ■



**Proof of Theorem 2.1:**

We now focus on the normality.

$$\begin{aligned}
& \sqrt{Nh} \left( \hat{\beta}(z) - \beta(z) \right) \\
&= \left( \frac{1}{N} \sum_{i=1}^N x_i x_i' K_h(z_i - z) \right)^{-1} \sqrt{\frac{h}{N}} \sum_{i=1}^N x_i x_i' (\beta(z_i) - \beta(z)) K_h(z_i - z) \\
&\quad + \left( \frac{1}{N} \sum_{i=1}^N x_i x_i' K_h(z_i - z) \right)^{-1} \sqrt{\frac{h}{N}} \sum_{i=1}^N x_i u_i K_h(z_i - z) \\
&:= A_1 + A_2.
\end{aligned} \tag{A.9}$$

By Lemma A.2, we just need to focus on  $\sqrt{\frac{h}{N}} \sum_{i=1}^N x_i u_i K_h(z_i - z)$ . Notice that by the proof for (2) of Lemma 2.2

$$\begin{aligned}
& \text{Var} \left[ \sqrt{\frac{h}{N}} \sum_{i=1}^N x_i u_i K_h(z_i - z) \right] \\
&= \frac{h}{N} \sum_{i_1=1}^N \sum_{i_2=1}^N E[x_{i_1} x_{i_2}' \sigma(x_{i_1}, z_{i_1}) \sigma(x_{i_2}, z_{i_2}) K_h(z_{i_1} - z) K_h(z_{i_2} - z)] \gamma_{i_1, i_2} \\
&= \frac{1}{N} \sum_{i_1=1}^N E \left[ \Sigma_{xx\sigma}(z) \frac{1}{h} K^2 \left( \frac{z_{i_1} - z}{h} \right) \right] B_{i_1} \\
&\quad + \frac{h}{N} \sum_{i_1, i_2=1, i_1 \neq i_2}^N E[\eta(z_{i_1}) K_h(z_{i_1} - z)] E[\eta(z_{i_2})' K_h(z_{i_2} - z)] \gamma_{i_1, i_2} \\
&= \frac{1}{N} \sum_{i_1=1}^N f_{i_1}(z) B_{i_1} \int \Sigma_{xx\sigma}(w) K^2(w) dw \\
&\quad + \frac{h}{N} \sum_{i_1, i_2=1, i_1 \neq i_2}^N f_{i_1}(z) f_{i_2}(z) \gamma_{i_1, i_2} \int \eta(w) K(w) dw \int \eta(w)' K(w) dw \\
&\quad + O(h^2) + O \left( \frac{\Delta_{2N}}{N^2} \right) + O \left( \frac{\Delta_{1N}}{Nh^2} \right) \\
&= V_1(z) + o(1),
\end{aligned} \tag{A.10}$$

where  $\Sigma_{xx\sigma}(z) = E[x_i x_i' \sigma(x_i, z_i) | z_i = z]$ ; the third equality follows from the procedure similar to (A.3) and (A.8).

Further write

$$\begin{aligned}
& \sqrt{\frac{h}{N}} \sum_{i=1}^N x_i u_i K_h(z_i - z) = \sqrt{\frac{h}{N}} \sum_{i=1}^N x_i \sigma(x_i, z_i) e_i K_h(z_i - z) \\
&= \sqrt{\frac{h}{N}} \sum_{i=1}^N x_i \sigma(x_i, z_i) K_h(z_i - z) \sum_{j=1}^{\infty} b_{ij} \varepsilon_j = \sum_{j=1}^N w_{jN} \varepsilon_j + \sum_{j=N+1}^{\infty} w_{jN} \varepsilon_j,
\end{aligned} \tag{A.11}$$

where  $w_{jN} = \sqrt{\frac{h}{N}} \sum_{i=1}^N x_i \sigma(x_i, z_i) K_h(z_i - z) b_{ij}$ . By the Cramer-Wold device, in order to derive asymptotic normality of the vector, we consider

$$\sqrt{\frac{h}{N}} \sum_{i=1}^N c' x_i u_i K_h(z_i - z) = \sum_{j=1}^N c' w_{jN} \varepsilon_j + \sum_{j=N+1}^{\infty} c' w_{jN} \varepsilon_j, \quad (\text{A.12})$$

where  $c \in \mathbb{R}^p$  is a fixed vector satisfying  $\|c\| = 1$ .

By (A.10), there must a sufficiently large  $M$  satisfying  $E \left[ \sum_{j=N+1}^{\infty} c' w_{jN} \varepsilon_j \right]^2 = o(1)$  for  $N > M$ . Since  $c' w_{jN} \varepsilon_j$  is martingale difference, we just need to focus on verifying the next two terms

$$\sum_{j=1}^N E [c' w_{jN} \varepsilon_j]^2 \rightarrow 1, \quad (\text{A.13})$$

$$\sum_{j=1}^N E \left[ (c' w_{jN} \varepsilon_j)^2 1(|c' w_{jN} \varepsilon_j| > \epsilon) \right] \rightarrow_P 0, \text{ for } \epsilon > 0. \quad (\text{A.14})$$

For (A.13), write

$$\sum_{j=1}^N E [c' w_{jN} \varepsilon_j]^2 = \sum_{j=1}^N (c' w_{jN})^2 - \sum_{j=N+1}^{\infty} (c' w_{jN})^2 = c' V_1(z_c, z_d) c + o(1).$$

Next let  $\nu$  be as in Assumption 1. Since  $\{x_i, i \geq 1\}$  and  $\{z_i, i \geq 1\}$  are independent of  $\{\varepsilon_j, j \geq 1\}$ , we then proceed further by conditional on  $\{x_i, i \geq 1\}$  and  $\{z_i, i \geq 1\}$ . Then, unconditionally, the results automatically hold. Conditional on  $\{x_i, i \geq 1\}$  and  $\{z_i, i \geq 1\}$ , we have

$$\begin{aligned} & \sum_{j=1}^N E [(c' w_{jN} \varepsilon_j)^2 1(|c' w_{jN} \varepsilon_j| > \epsilon)] = \sum_{j=1}^N (c' w_{jN})^2 E [\varepsilon_j^2 1(|c' w_{jN} \varepsilon_j| > \epsilon)] \\ & \leq \sum_{j=1}^N (c' w_{jN})^2 \{E [|\varepsilon_j|^{2+\nu}] \}^{\frac{2}{2+\nu}} \{E [1(|c' w_{jN} \varepsilon_j| > \epsilon)] \}^{\frac{\nu}{2+\nu}} \\ & \leq \sum_{j=1}^N (c' w_{jN})^2 \{E [|\varepsilon_j|^{2+\nu}] \}^{\frac{2}{2+\nu}} \left\{ \frac{E[|c' w_{jN} \varepsilon_j|]}{\epsilon} \right\}^{\frac{\nu}{2+\nu}} \\ & \leq \sum_{j=1}^N (c' w_{jN})^2 \{E [|\varepsilon_j|^{2+\nu}] \}^{\frac{2}{2+\nu}} \left\{ \frac{|c' w_{jN}|}{\epsilon} \right\}^{\frac{\nu}{2+\nu}} \{E |\varepsilon_j| \}^{\frac{\nu}{2+\nu}} \\ & = \sum_{j=1}^N |c' w_{jN}|^{2+\frac{\nu}{2+\nu}} \epsilon^{-\frac{\nu}{2+\nu}} \{E [|\varepsilon_j|^{2+\nu}] \}^{\frac{2}{2+\nu}} \{E |\varepsilon_j| \}^{\frac{\nu}{2+\nu}} \\ & \leq O(1) \left\{ \max_{1 \leq j \leq N} |c' w_{jN}|^{\frac{\nu}{2+\nu}} \right\} \sum_{j=1}^N |c' w_{jN}|^2 \epsilon^{-\frac{\nu}{2+\nu}}. \end{aligned}$$

We then just need to verify that  $\max_{1 \leq j \leq n(N)} |c' w_{jN}|^{\frac{\nu}{2+\nu}} \rightarrow 0$ . We then obtain

$$\begin{aligned} |c' w_{jN}| &= \left| c' \sqrt{\frac{h}{N}} \sum_{i=1}^N x_i \sigma(x_i, z_i) K_h(z_i - z) b_{ij} \right| \\ &\leq \|c\| \sqrt{\frac{h}{N}} \sum_{i=1}^N \|x_i \sigma(x_i, z_i) K_h(z_i - z) b_{ij}\| \\ &\leq O(1) \sqrt{\frac{1}{Nh}} \sum_{i=1}^N |b_{ij}| \leq O(1) \max_{1 \leq j \leq N} \frac{1}{\sqrt{Nh}} \sum_{i=1}^N |b_{ij}| \rightarrow 0. \end{aligned}$$

Then the proof is complete. ■

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